

OPTION SPANNING BEYOND L_p -MODELS

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ABSTRACT. The aim of this paper is to study the spanning power of options in a static financial market that allows non-integrable assets. Our findings extend and unify the results in [13, 14, 24] for L_p -models. We also apply the spanning power properties to the pricing problem. In particular, we show that prices on call and put options of a limited liability asset can be uniquely extended by arbitrage to all marketed contingent claims written on the asset.

1. INTRODUCTION

Throughout this paper, Ω stands for the state space of a financial market, Σ stands for the σ -algebra modelling the market information structure, and \mathbb{P} stands for a probability over (Ω, Σ) . The space of contingent claims, X , is modelled as an ideal (i.e., solid subspace) of $L_0(\Sigma)$ containing the constant functions, which represent investments in the riskless asset. A claim displays limited liabilities if it is positive. For a limited liability claim f , its option space is the collection of all portfolios of call and put options written on f , which can be identified as follows:

$$O_f = \text{Span} \{1, (f - k)^+ : k \in \mathbb{R}\}.$$

The space of all contingent claims written on f is identified as the space of all functions measurable with respect to $\sigma(f)$, the sub- σ -algebra

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generated by f , i.e.,

$$L_0(\sigma(f)).$$

A stream of research has been devoted to the study of spanning power of options on f , i.e., the size of O_f . In the seminal paper [30], Ross showed that if the state space Ω is finite then the options on f span the space of contingent claims written on f , i.e.,

$$O_f = L_0(\sigma(f)),$$

and if, in addition, f is one-to-one, then the option space of f completes the market, i.e.,

$$O_f = L_0(\Sigma).$$

These elegant results of Ross have inspired many successive contributions to the study of options. See e.g. [7, 20, 22] for related results on finite state spaces. In particular, they have also been examined for financial markets with infinite state spaces

Nachman proved in [24] that if $X = L_p(\Sigma)$ ($1 \leq p < \infty$), then the options on f span the space of contingent claims written on the asset in two ways: approximating by a.e. convergence or by p -th mean convergence. Precisely, it was proved that an asset $x \in L_p(\Sigma)$ is a contingent claim on $f \in L_p(\Sigma)$ iff there exists a sequence of portfolios of options on f converging a.e. to x iff there exists a sequence of portfolios of options on f converging in the p -th mean to x . That is,

$$\overline{O_f}^{a.e.} \cap L_p(\Sigma) = \overline{O_f}^{\|\cdot\|_p} = L_0(\sigma(f)) \cap L_p(\Sigma). \quad (1)$$

Galvani ([13]) and Galvani and Troitsky ([14]) proved further that if Ω is a Polish space equipped with the Borel σ -algebra and f is one-to-one and bounded, then O_f completes the market $X = L_p(\Sigma)$ ($1 \leq p \leq \infty$). That is, for $1 \leq p < \infty$,

$$\overline{O_f}^{a.e.} \cap L_p(\Sigma) = \overline{O_f}^{\|\cdot\|_p} = L_p(\Sigma), \quad (2)$$

and

$$\overline{O_f}^{a.e.} \cap L_\infty(\Sigma) = \overline{O_f}^{w^*} = L_\infty(\Sigma). \quad (3)$$

In this paper, we explore the spanning power of options in general spaces of contingent claims. Our contributions here are two-fold. Firstly, the spaces of contingent claims in our setting can be modelled as any ideal of $L_0(\Sigma)$ which contains the constant functions and admits a strictly positive order continuous linear functional. This framework includes not only the L_p -space ($1 \leq p \leq \infty$) models, but also the much wider class of Orlicz space models as well as many non-integrable space models which have been extensively used in the theory of risk measures (see e.g. [5, 6, 9, 11, 16, 25]).

Secondly, we provide an approach to unify the norm and w^* -topologies used in the results of Nachman, Galvani and Troitsky, and thus give more comprehensive insight into the general structures of option spaces; see Theorem 3.1 and Remark 3.3. The unification in our approach is due to the use of the topology $\sigma(X, X_n^\sim)$, where X_n^\sim is the set of all order continuous linear functionals on X .

Observe that $(X_n^\sim)_+$ has a natural connection with linear pricing functionals. Recall that a linear pricing functional ϕ on X is given by a state-price density $y \geq 0$ via integration, i.e.

$$\phi(x) = \int_{\Omega} xy \, d\mathbb{P} \text{ for all } x \in X,$$

where y is measurable and satisfies $\int_{\Omega} |xy| \, d\mathbb{P} < \infty$ for all $x \in X$. By Dominated Convergence Theorem, it is easily seen that ϕ is order continuous on X . Conversely, by Radon-Nikodym theorem, one can easily see that each positive order continuous linear functional on X has a positive density, and thus is a linear pricing functional. Therefore, $(X_n^\sim)_+$ is just the collection of linear pricing functionals on X .

Because of this link, we are able to apply Theorem 3.1 and shed light on the following general problem, raised in [8]: “Under what circumstances can prices on the marketed assets or basic derivative assets be uniquely extended by arbitrage to prices on all derivative assets in a large class and when is such an extension unique?” In Theorem 3.4, we prove that when the arbitrage condition is understood

as a no-free lunch condition (see [21]), one can extend uniquely the prices on O_f to the marketed contingent claims written on f .

Finally, we mention that there is a stream of works studying market completion using options in a continuous time setting. In this framework, the model is said to be complete, if any contingent claim payoff can be obtained as the terminal value of a self-financing trading strategy. We refer the reader to the recent papers [10, 18, 26, 29, 32].

2. PRELIMINARY RESULTS

We refer to [3, 4] for all unexplained terminology and standard facts on vector and Banach lattices. A vector subspace Y of a vector lattice X is called a **sublattice** if $|y| \in Y$ whenever $y \in Y$; in this case, $y_1 \wedge y_2$ and $y_1 \vee y_2$ both belong to Y whenever $y_1, y_2 \in Y$. A subspace Y is called an **ideal** (or a **solid subspace**) of X , if $|x| \leq |y|$ and $y \in Y$ imply $x \in Y$. A linear functional ϕ on a vector lattice X is said to be **order continuous** if $\phi(x_\alpha) \rightarrow 0$ whenever $x_\alpha \xrightarrow{o} 0$ in X . The collection of all order continuous linear functionals on X is denoted by X_n^\sim and is called the **order continuous dual** of X . A linear functional ϕ on X is said to be **positive** if $\phi(x) \geq 0$ whenever $x \geq 0$, and is said to be **strictly positive** if $\phi(x) > 0$ whenever $x > 0$.

The following lemma will be used. Recall first that a vector lattice is said to be **order complete** (or **Dedekind complete**) if every order bounded above subset has a supremum, and is said to have the **countable sup property** if any subset having a supremum possesses a countable subset with the same supremum. A subset A of a vector lattice X is said to be **order closed** if $x \in A$ whenever there exists a net (a_α) in A such that $a_\alpha \xrightarrow{o} x$ in X .

Lemma 2.1. *Let X be an order complete vector lattice with the countable sup property and Y be a sublattice of X . Then Y is order closed in X iff for any increasing sequence in Y which is order bounded above in X , its supremum in X also lies in Y .*

Given a probability space $(\Omega, \Sigma, \mathbb{P})$, denote by $L_0(\Sigma)$ the space of all real-valued measurable functions (*modulo a.e. equality*). We use 1 to denote the constant one function. Recall that $L_0(\Sigma)$ is a vector lattice, endowed with the natural order: $f \leq g$ iff $f(\omega) \leq g(\omega)$ for a.e. $\omega \in \Omega$. By [23, Lemma 2.6.1], it is easily seen that any ideal of $L_0(\Sigma)$ is order complete and has the countable sup property. Hence, Lemma 2.1 is applicable to them. Recall also that $f_n \xrightarrow{o} 0$ in an ideal X of $L_0(\Sigma)$ if and only if $f_n \xrightarrow{a.e.} 0$ and $(f_n)_{n=1}^\infty$ is *order bounded* in X , i.e., there exists $f \in X$ such that $|f_n| \leq f$ a.e. for each $n \geq 1$. We remark that the class of ideals of $L_0(\Sigma)$ which admit strictly positive order continuous linear functionals is very large. For example, by [15, Proposition 5.19], all Banach function spaces (i.e., ideals of $L_0(\Sigma)$ endowed with complete lattice norm), including all Orlicz spaces, are as such.

For a subset Y of $L_0(\Sigma)$, define $\sigma(Y)$ to be the smallest sub- σ -algebra of Σ which makes all members in Y measurable and contains all \mathbb{P} -null sets. Denote by $L_0(\sigma(Y))$ the set of all functions in $L_0(\Sigma)$ which are measurable with respect to $\sigma(Y)$. Clearly, $Y \subset L_0(\sigma(Y))$. If $Y = \{f\}$, we write $\sigma(f)$ instead of $\sigma(\{f\})$, for the sake of simplicity. The following result is an improved and generalized market completeness theorem in the sense of Green and Jarrow ([17, Theorem 1]).

Lemma 2.2. *Let X be an ideal of $L_0(\Sigma)$ and Y be a sublattice of X such that $1 \in Y$. Then the following are equivalent:*

- (a) Y is order closed in X ,
- (b) $Y = L_0(\sigma(Y)) \cap X$.

3. MAIN RESULTS

In this section, the space of contingent claims, X , is always modelled as an ideal of $L_0(\Sigma)$ over a given probability space $(\Omega, \Sigma, \mathbb{P})$ that contains the constant functions and admits a strictly positive order continuous linear functional. Our first main result is as follows.

Theorem 3.1. *Let f be a limited liability claim in X . For a claim $g \in X$, the following are equivalent:*

- (a) *g is a contingent claim written on f , i.e., $g \in L_0(\sigma(f)) \cap X$,*
- (b) *g can be approximated by portfolios of options on f in the $\sigma(X, X_n^\sim)$ -topology, i.e., $g \in \overline{O_f}^{\sigma(X, X_n^\sim)}$,*
- (c) *There exists a sequence (g_n) in O_f such that $g_n \xrightarrow{a.e.} g$.*

The following corollary is immediate.

Corollary 3.2. *Let f be a limited liability claim in X such that $\sigma(f) = \Sigma$. Then we have the following:*

- (a) *The option space of f completes the market in the $\sigma(X, X_n^\sim)$ -topology, i.e., $\overline{O_f}^{\sigma(X, X_n^\sim)} = X$,*
- (b) *The option space of f completes the market by approximating via a.e. convergence, i.e., for any $g \in X$, there exists a sequence (g_n) in O_f such that $g_n \xrightarrow{a.e.} g$.*

Remark 3.3. Note that our Theorem 3.1 and Corollary 3.2 imply both the aforementioned results of Nachman, Galvani and Troitsky. Indeed, recall first that, for $1 \leq p < \infty$, $X = L_p(\Sigma)$ is **order continuous**, i.e., $x_\alpha \downarrow 0$ in X implies $\|x_\alpha\| \downarrow 0$. In this case, one has $X_n^\sim = X^*$, so that $\sigma(X, X_n^\sim)$ is just the weak topology on X . Thus, by Mazur's theorem, $\overline{C}^{\sigma(X, X_n^\sim)} = \overline{C}^w = \overline{C}^{\|\cdot\|}$ for any convex subset C of X . Consequently, it follows that

$$\overline{O_f}^{\sigma(X, X_n^\sim)} = \overline{O_f}^{\|\cdot\|_p}.$$

Now it is clear that Equation (1) follows from Theorem 3.1. If, in addition, Ω is a Polish space with Σ being the Borel algebra, and f is one-to-one, then it is easily seen that $\sigma(f) = \Sigma$ by [4, Theorem 12.29]. Thus, Equation (2) follows from Corollary 3.2. Equation (3) also follows from Corollary 3.2, since $L_\infty(\Sigma)_n^\sim = L_1(\Sigma)$ and thus $\sigma(L_\infty(\Sigma), L_\infty(\Sigma)_n^\sim)$ is just the w^* -topology.

We now turn to discuss the pricing problem. Our notation and terminology are in accordance with [21, 31].

Let f be a fixed asset in X and π be a positive linear functional on the option space $M := O_f$, which is interpreted as a linear pricing functional on M . We denote by $M_0 := \{x \in M \mid \pi(x) = 0\}$ the set of all portfolios of options on f that can be bought or sold with zero price. We say that (M, π) admits **no free lunches** (cf. [31, Definition 1.3]), if the following holds

$$C \cap X_+ = \{0\}, \text{ where } C = \overline{M_0 - X_+}^{\sigma(X, X_n^\sim)}.$$

We say that a price, p , of an asset $g \in X$ is **consistent** with (M, π) if there exists a strictly positive functional $x^* \in X_n^\sim$ such that $x^*|_M = \pi$ and $x^*(g) = p$ ([21, Definition, pp. 29]). The price of $g \in X$ is said to be **determined by arbitrage from** (M, π) if there is a single price p for g that is consistent with (M, π) ([21, Definition, pp. 30]).

Theorem 3.4. *Suppose that the space X of contingent claims is a Banach function space in $L_0(\Sigma)$. Let f be a limited liability asset in X and π be a positive linear functional on the option space $M = O_f$. If (M, π) admits no free lunches, then the price of any contingent claim $g \in L_0(\sigma(f)) \cap X$ is determined by arbitrage from (M, π) .*

The proof of this result essentially depends on the following version of the Kreps-Yan Theorem, which is of independent interest.

Proposition 3.5. *Let X be a Banach function space in $L_0(\Sigma)$. Then the Kreps-Yan theorem holds true for $(X, \sigma(X, X_n^\sim))$. That is, for each $\sigma(X, X_n^\sim)$ -closed cone C in X such that $C \supset -X_+$ and $C \cap X_+ = \{0\}$, there exists a strictly positive functional $\phi \in X_n^\sim$ such that $\phi|_C \leq 0$.*

The proof of this result (see Section 4) relies on [19, Theorem 3.1]. For more results in this direction, we refer the reader to [27, 28]. For no-arbitrage results, we refer the reader to the monograph [12] and the references therein.

4. PROOFS OF RESULTS

Proof of Lemma 2.1. Let (y_n) be an increasing sequence in Y that is order bounded above in X . Since X is order complete, it follows that (y_n) has a supremum, x , in X . Since (y_n) is increasing, it follows that $y_n \uparrow x$ in X , so that $y_n \xrightarrow{o} x$ in X . Thus, if Y is order closed in X , then $x \in Y$. This proves the “only if” part.

For the “if” part, observe first that, in this case, for any sequence (y_n) in Y which is *order bounded* in X , its supremum and infimum in X also lie in Y . Indeed, denote by x the supremum of (y_n) in X . Put $z_n = \bigvee_{k=1}^n y_k$. Then $z_n \in Y$ as Y is a sublattice of X , and moreover, the supremum of (z_n) in X is still x . Since (z_n) is increasing, it follows from the “if” assumption that $x \in Y$. Replacing (y_n) with $(-y_n)$, one sees easily that the infimum of (y_n) in X also lies in Y . Now let $(y_\alpha) \subset Y$ and $x \in X$ be such that $y_\alpha \xrightarrow{o} x$ in X . By passing to a tail, we may assume that (y_α) is order bounded in X . Then since X is order complete, we have

$$\inf_{\alpha} \sup_{\beta \geq \alpha} |y_\beta - x| = 0,$$

where all the suprema and infima are taken in X . By the countable sup property of X , we can choose $\{\alpha_n\}_{n=1}^\infty$ such that $\inf_n \sup_{\beta \geq \alpha_n} |y_\beta - x| = 0$. Without loss of generality, we can assume that (α_n) is increasing. It follows that

$$\inf_n \sup_{m \geq n} |y_{\alpha_m} - x| = 0,$$

or equivalently, $y_{\alpha_n} \xrightarrow{o} x$, so that $x = \inf_n \sup_{m \geq n} y_{\alpha_m}$; cf. [4, Theorem 8.16]. Applying the preceding observation to the suprema and then to the infimum, we obtain that $x \in Y$. \square

Proof of Lemma 2.2. Assume first (b) holds. Let (f_n) be an increasing sequence in Y and f be its supremum in X . Then $f_n \uparrow f$ a.e. Since each f_n is $\sigma(Y)$ -measurable, we have that f is also $\sigma(Y)$ -measurable, so that $f \in L_0(\sigma(Y)) \cap X = Y$. Thus since X is order complete and has the countable sup property, Lemma 2.1 implies that (a) holds.

Conversely, assume that (a) holds. We first claim that $\sigma(Y) = \{A \in \Sigma : \chi_A \in Y\}^1$. Denote the right hand side by \mathcal{G} . We first show that it is a σ -algebra. Indeed, it is clear that $\emptyset \in \mathcal{G}$, and that if $A \in \mathcal{G}$, then $\chi_{A^c} = 1 - \chi_A \in Y$, so that $A^c \in \mathcal{G}$. Now let $(A_k)_{k=1}^\infty$ be a sequence of sets in \mathcal{G} . Then $\chi_{\bigcup_{k=1}^n A_k} = \bigvee_{k=1}^n \chi_{A_k} \in Y$, and from $\chi_{\bigcup_{k=1}^n A_k} \uparrow \chi_{\bigcup_{k=1}^\infty A_k}$ in X , it follows that $\chi_{\bigcup_{k=1}^\infty A_k} \in Y$, since Y is order closed. Therefore, $\bigcup_{k=1}^\infty A_k \in \mathcal{G}$. This concludes the proof of that \mathcal{G} is a σ -algebra. Next, we show that each $f \in Y$ is measurable with respect to \mathcal{G} . Indeed, for any real number r , it follows from $Y \ni n(f - r)^+ \wedge 1 \uparrow \chi_{\{f > r\}}$ in X that $\chi_{\{f > r\}} \in Y$, so that $\{f > r\} \in \mathcal{G}$, and f is \mathcal{G} -measurable. Clearly, \mathcal{G} contains all \mathbb{P} -null sets². Thus we have $\sigma(Y) \subset \mathcal{G}$. The reverse inclusion being clear, this completes the proof of the claim.

It is clear that $Y \subset L_0(\sigma(Y)) \cap X$. Now take $f \in L_0(\sigma(Y)) \cap X$. By considering f^\pm , we may assume that f is non-negative. Then we can find a sequence (f_n) of simple functions which are measurable with respect to $\sigma(Y)$ such that $f_n \uparrow f$ everywhere, so that $f_n \uparrow f$ in X . By the preceding claim, we have that $f_n \in Y$. Therefore, $f \in Y$, and thus $L_0(\sigma(Y)) \cap X \subset Y$. It follows that $Y = L_0(\sigma(Y)) \cap X$. \square

Proof of Theorem 3.1. We claim that O_f is a sublattice of X . Indeed, put $Z = \text{Span}\{b, s, (s - kb)^+ : k \in \mathbb{R}\}$, where $b = f + 1$ and $s = f$. Note that, being an ideal of $L_0(\Sigma)$, X is order complete, and thus it is uniformly complete (cf. [2, Lemma 1.56]). By [8, Theorem (1)], it follows that Z is a sublattice of X . Now simply observe that $Z = O_f$. Indeed, the inclusion $O_f \subset Z$ is immediate as $f, 1 \in Z$ and Z is closed under lattice operations. For the reverse inclusion, note that $s = f = (f - 0)^+ \in O_f$ so that $b \in O_f$ as well. Also, for $k \geq 1$ we have $(s - kb)^+ = 0$, and for $k < 1$ we have $(s - kb)^+ = (1 - k)(f - \frac{k}{1 - k}1)^+ \in O_f$.

Assume that (c) holds. Since Z is a sublattice of X , by considering the positive and negative parts, respectively, we may assume that $g \geq 0$ and $g_n \geq 0$ for all n . For any $k \geq 1$, since $g_k \wedge g_n \xrightarrow{a.e.} g_k \wedge g$ and

¹This is essentially contained in [17, Theorem 1].

²Keep in mind that χ_A is identified as 0 in $L_0(\Sigma)$ if $\mathbb{P}(A) = 0$.

$(g_k \wedge g_n)_n$ is order bounded in X , it follows that $g_k \wedge g_n \xrightarrow{o} g_k \wedge g$ in X , and therefore, $g_k \wedge g_n \xrightarrow{\sigma(X, X_n^\sim)} g_k \wedge g$ as $n \rightarrow \infty$. By the fact that Z is a sublattice again, we have $g_k \wedge g_n \in Z$ for all $k, n \geq 1$. Hence,

$$g_k \wedge g \in \overline{Z}^{\sigma(X, X_n^\sim)}$$

for any $k \geq 1$. Now $g_k \wedge g \xrightarrow{a.e.} g$ and $(g_k \wedge g)$ is order bounded in X , we have $g_k \wedge g \xrightarrow{o} g$ in X , so that $g_k \wedge g \xrightarrow{\sigma(X, X_n^\sim)} g$. Therefore, since $\overline{Z}^{\sigma(X, X_n^\sim)}$ is $\sigma(X, X_n^\sim)$ -closed, we have

$$g \in \overline{Z}^{\sigma(X, X_n^\sim)}.$$

This proves that (c) \Rightarrow (b).

Suppose now (b) holds. Recall that X_n^\sim is a band (i.e., order closed ideal) of the order dual X^\sim ([3, Theorem 1.57]). It follows from [3, Theorem 3.50] that the dual of X under the topology $|\sigma|(X, X_n^\sim)$ is just X_n^\sim . Therefore, by Mazur's theorem (cf. [3, Theorem 3.13]), since Z is convex, $\overline{Z}^{\sigma(X, X_n^\sim)} = \overline{Z}^{|\sigma|(X, X_n^\sim)}$. Consequently, there exists a net (g_α) in Z such that $g_\alpha \xrightarrow{|\sigma|(X, X_n^\sim)} g$. In particular, if x_0^* is any strictly positive order continuous functional on X , then

$$x_0^*(|g_\alpha - g|) \rightarrow 0.$$

Take (α_n) such that $x_0^*(|g_{\alpha_n} - g|) \leq \frac{1}{2^n}$. Then since $\bigvee_{m=n}^k |g_{\alpha_m} - g| \wedge 1 \uparrow_k \sup_{m \geq n} |g_{\alpha_m} - g| \wedge 1$, it follows from order continuity of x_0^* that

$$\begin{aligned} x_0^*\left(\sup_{m \geq n} |g_{\alpha_m} - g| \wedge 1\right) &= \lim_k x_0^*\left(\bigvee_{m=n}^k |g_{\alpha_m} - g| \wedge 1\right) \\ &\leq \lim_k x_0^*\left(\sum_{m=n}^k |g_{\alpha_m} - g| \wedge 1\right) \leq \frac{1}{2^{n-1}}. \end{aligned}$$

Therefore, we have

$$x_0^*\left(\inf_{n \geq 1} \sup_{m \geq n} |g_{\alpha_m} - g| \wedge 1\right) = 0,$$

and thus by strict positivity of x_0^* , we have

$$\inf_{n \geq 1} \sup_{m \geq n} |g_{\alpha_m} - g| \wedge 1 = 0.$$

If $g_{\alpha_n} \not\xrightarrow{a.e.} g$, then there exist $\varepsilon > 0$ and a measurable set A of positive measure such that $\limsup_n |g_n(\omega) - g(\omega)| \geq \varepsilon$ for any $\omega \in A$. Therefore, it is easily seen that

$$\inf_{n \geq 1} \sup_{m \geq n} |g_{\alpha_n} - g| \wedge 1 \geq (\varepsilon \chi_A) \wedge 1 > 0.$$

This contradiction concludes the proof of (b) \Rightarrow (c).

Now put $Y = \overline{Z}^{\sigma(X, X_n^\sim)}$. Then Y is clearly order closed in X . Moreover, by the preceding paragraph, $Y = \overline{Z}^{|\sigma|(X, X_n^\sim)}$, implying that it is also a sublattice of X by [3, Theorem 3.46]. Thus by Lemma 2.2, $Y = L_0(\sigma(Y)) \cap X$. Since $f \in Y$, it is clear that $\sigma(Y) \supset \sigma(f)$, so that

$$Y = L_0(\sigma(Y)) \cap X \supset L_0(\sigma(f)) \cap X.$$

For the reverse inclusion, note that, by definition of O_f , it is easily seen that each $g \in O_f$ is measurable with respect to $\sigma(f)$. Now for an arbitrary $g \in Y$, we can take, by the implication (b) \Rightarrow (c), a sequence (g_n) in O_f such that $g_n \xrightarrow{a.e.} g$. Clearly, g is also $\sigma(f)$ -measurable. Therefore, it follows that

$$Y \subset L_0(\sigma(f)) \cap X,$$

and hence $Y = L_0(\sigma(f)) \cap X$. This proves (a) \Leftrightarrow (b). \square

Proof of Proposition 3.5. We apply [19, Theorem 3.1] to $(X, \sigma(X, X_n^\sim))$, and verify that the following Assumptions (C) and (L) are satisfied.

Assumption (C): For every sequence (x_n^*) in X_n^\sim , there exist strictly positive numbers (α_n) such that $\sum_{n=1}^\infty \alpha_n x_n^*$ converges in X_n^\sim with respect to the $\sigma(X_n^\sim, X)$ -topology.

Assumption (L): Any family $\{x_\gamma^*\}_{\gamma \in \Gamma}$ in $(X_n^\sim)_+$ admits a countable subfamily $\{x_{\gamma_n}^*\}_{n \geq 1}$ such that, for any $x \in X_+$, $x_{\gamma_n}^*(x) = 0$ for all $n \geq 1$ implies $x_\gamma^*(x) = 0$ for all $\gamma \in \Gamma$.

We first verify that Assumption (C) is satisfied. Indeed, since X is a Banach lattice, we know that the order dual X^\sim equals the norm dual X^* ([3, Corollary 4.5]) and is thus a Banach lattice. By [3, Theorem 1.57], X_n^\sim is a band (i.e., order closed ideal) in $X^\sim = X^*$, and

is thus norm closed in X^* by [3, Theorem 3.46]. Now for a sequence (x_n^*) in X_n^\sim , put $\alpha_n = \frac{1}{2^n \|x_n^*\| + 1}$ for each $n \geq 1$. Then α_n 's are strictly positive, and $\sum_1^\infty \alpha_n x_n^*$ converges in norm to some $x^* \in X^*$. Since X_n^\sim is norm closed in X^* , it follows that $x^* \in X_n^\sim$. Clearly, $\sum_1^\infty \alpha_n x_n^*$ also converges to x^* in the $\sigma(X_n^\sim, X)$ -topology.

We now verify that Assumption (L) is also satisfied. For the given family $\{x_\gamma^*\}_{\gamma \in \Gamma}$ in $(X_n^\sim)_+$, put $N_\gamma := \{x \in X : x_\gamma^*(|x|) = 0\}$ and $C_\gamma := N_\gamma^d := \{x \in X : |x| \wedge |y| = 0 \text{ for all } y \in N_\gamma\}$ for each γ . Observe that N_γ is a band. Indeed, it is clearly an ideal. If a net (x_α) in N_γ converges in order to some $x \in X$, then $|x_\alpha - x| \xrightarrow{o} 0$ implies that $x_\gamma^*(|x|) = |x_\gamma^*(|x_\alpha|) - x_\gamma^*(|x|)| \leq x_\gamma^*(||x_\alpha| - |x||) \leq x_\gamma^*(|x_\alpha - x|) \rightarrow 0$, and consequently, $x_\gamma^*(|x|) = 0$, i.e., $x \in N_\gamma$. This yields the *band decomposition* $X = N_\gamma \oplus C_\gamma$ by [3, Theorem 1.42]. Recall from [1, Corollary 5.22] that X has a weak unit $u > 0$, i.e., any function $x \in X$ is supported in $\{\omega : u(\omega) > 0\}$ off a null set. Write $u = f_\gamma + e_\gamma$ where $f_\gamma \in N_\gamma$ and $e_\gamma \in C_\gamma$. Since $f_\gamma \wedge e_\gamma = 0$, it is easily seen that there exists $A_\gamma \in \Sigma$ such that $e_\gamma = u\chi_{A_\gamma}$ and $f_\gamma = u\chi_{A_\gamma^c}$. Each function in N_γ is disjoint with e_γ and is thus supported in A_γ^c off a null set; each function in C_γ is disjoint with f_γ and is thus supported in A_γ off a null set. By countable sup property of X , we choose $\{\gamma_n\}_{n=1}^\infty$ such that $\sup_n e_{\gamma_n} = \sup_\gamma e_\gamma$. If $x_{\gamma_n}^*(x) = 0$ for all $n \geq 1$ and some $x \in X_+$, then we have $x \in N_{\gamma_n}$, so that $x \wedge e_{\gamma_n} = 0$, for all $n \geq 1$. It follows that $x \wedge \sup_\gamma e_\gamma = x \wedge \sup_n e_{\gamma_n} = \sup_n (x \wedge e_{\gamma_n}) = 0$, and consequently, $x \wedge e_\gamma = 0$ for any γ . This implies that x is supported in A_γ^c off a null set and hence belongs to N_γ , i.e., $x_\gamma^*(x) = 0$. \square

Proof of Theorem 3.4. It is clear that $C := \overline{M_0 - X_+}^{\sigma(X, X_n^\sim)}$ is a $\sigma(X, X_n^\sim)$ -closed cone with $-X_+ \subset C$ and $C \cap X_+ = \{0\}$ because of no free lunches. Thus by Proposition 3.5, there exists a strictly positive linear functional $x^* \in X_n^\sim$ such that $x^*|_C \leq 0$. This last condition implies that $x^*|_{M_0} = 0$, so that $\ker \pi = M_0 \subset \ker(x^*|_M)$. Hence, there exists $\lambda > 0$ such that $\pi = \lambda x^*|_M$. Therefore, for each $g \in L_0(\sigma(f)) \cap X =$

$\overline{O}_f^{\sigma(X, X_n^\sim)}$, it is easily seen that the price $p := \lambda x^*(g)$ is consistent with (M, π) . \square

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